Physics Notes

BY

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Class:10+1
Unit: V
Topic: Motion of System of Particles and Rigid Body

<u>*SYLLABUS*</u>: UNIT-V

Centre of mass of a two particle system, momentum conversion and centre of mass motion. Centre of mass of a rigid body; centre of mass of uniform rod.
Vector product of vectors; moment of a force, torque, angular momentum, conservation of angular momentum with some examples.
Equilibrium of rigid bodies, rigid body rotation and equations of rotational motion, comparison of linear and rotational motions; moment of inertia, radius of gyration.
Values of moments of inertia for simple geometrical objects (no derivation). Statement of parallel and perpendicular axes theorem and their application.

Q.1. Derive an expression for K.E of rotation of an object?

Ans. A rigid boy revolves around an axis of rotation as shown in Fig. Different particles of body also move in circular motion.

Mass m_1, m_2, m_3 -------------- m_n move in circles

of radius r_1, r_2, r_3 ------------------- r_n with

velocity v_1, v_2, v_3 ----------------- v_n

K.E. Total $= K.E_1 + K.E_2 +$ --------------- $+ K.E_n$

$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 +$ ---------------- $+ \frac{1}{2} m_n v_n^2$

$= \frac{1}{2} m_1 (r_1 w)^2 + \frac{1}{2} m_2 (r_2 w)^2 +$ ------------------ $+ \frac{1}{2} m_n (r_n w)^2$

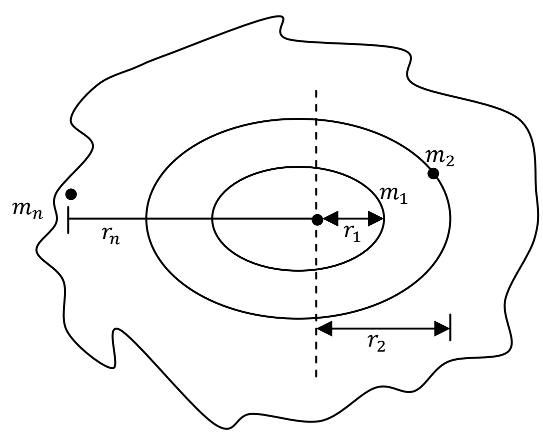
$= \frac{1}{2} m_1 r_1^2 w^2 + \frac{1}{2} m_2 r_2^2 w^2 +$ ------------------ $+ \frac{1}{2} m_n r_n^2 w^2$

$= \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2 + ----- + m_n r_n^2) w^2$

$= \frac{1}{2} (\Sigma m r^2) w^2$

$\boxed{\text{K.E. Total} = \frac{1}{2} \mathrm{I} w^2}$ Where I is moment of inertia.

$\mathrm{I} = \Sigma m r^2$



Q2. **a) Define moment of inertia? Factors on which moment of inertia depends? Physical significance?**
b) Units? Dimensional formula? Scalar, vector or something else? Discuss.

Ans.a) $\Sigma\, m\, r^2$ i.e. $m_1\, r_1^2 + \, m_2\, r_2^2 + \, ------ \, + \, m_n\, r_n^2$ is M.O.I.

Moment of inertia is defined as the sum of the products of masses of all the particles of the body and the square of their respective perpendicular distance from the axis of rotation.

<u>Moment of Inertia depends upon</u>:-

1. Position of the axis of rotation,
2. Orientation of the axis of rotation,
3. Shape of the body,
4. Size of the body,
5. Distribution of mass of the body about the axis of rotation.

<u>Physical Significance</u>:-

K.E $= \frac{1}{2}\, m\, v^2$ K.E $= \frac{1}{2}\, I\, w^2$

So "*mass*" in linear is analogue to "*Moment of Inertia*" in rotation i.e. Moment of Inertia is inertia of rotation.

b) **<u>Units</u>**:

M.O.I. $= m\, r^2$

S.I. Units = Kg m^2

cgs Units = gm cm^2

<u>Dimensional Formula</u>:

I $= [M][R^2]$

$= M^1\, L^2$

[I] $= [M^1\, L^2\, T^0]$

<u>Scalar/Vector</u>:

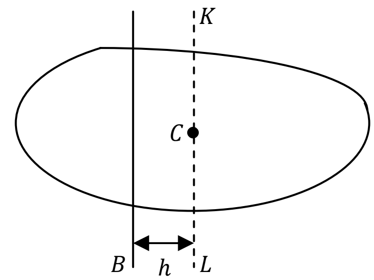
Neither scalar nor vector, It is a "*TENSOR*".

Q3. State, explain and prove

a) Parallel axis theorem

b) Perpendicular axis theorem.

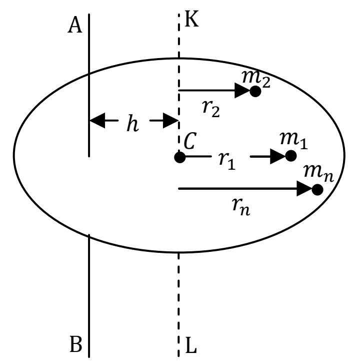
Ans.a) **<u>Parallel Axis Theorem</u>**:



$$\boxed{I_{AB} = I_{KL} + M\,h^2}$$

According to this theorem, moment of inertia of a rigid body about any axis *AB* is equal to moment of inertia of the body about another axis *KL* passing through centre of mass *C* of the body in a direction parallel to *AB*, plus the product of total mass *M* of the body and square of the perpendicular distance between two parallel area

<u>Proof</u>:



$$I_{AB} = m_1\,(r_1 + h)^2 + m_2\,(r_2 + h)^2 + \text{----------------} + m_n\,(r_n + h)^2$$

$$= \boxed{\begin{array}{l} m_1\,r_1^2 \\ + m_1\,h^2 \\ + 2\,m_1\,r_1\,h \end{array}} + \boxed{\begin{array}{l} m_2\,r_2^2 \\ + m_2\,h^2 \\ + 2\,m_2\,r_2 \end{array}} + \text{---------} + \boxed{\begin{array}{l} m_n\,r_n^2 \\ + m_n\,h^2 \\ + 2\,m_n\,r_n\,h \end{array}}$$

$$= (m_1\,r_1^2 + m_2\,r_2^2 + ------ + m_n\,r_n^2) +$$

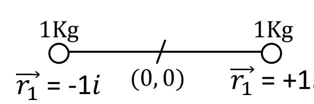
$$(m_1 + m_2 + ------- + m_n)h^2 +$$

$$2\,h\,(m_1\,\vec{r_1} + m_2\,\vec{r_2} + ------ + m_n\,\vec{r_n})$$

$$= \Sigma\,m_i\,r_i^2 + h^2\,\Sigma\,m_i + 2\,h\,(m_1\,\vec{r_1} + m_2\,\vec{r_2} + --$$

$$= I_{KL} + M\,h^2 + 0$$

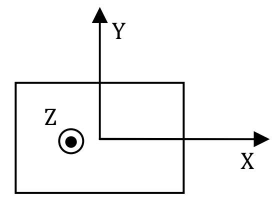
$\Sigma\,m_i\,\vec{r_i} = 0$ about C.O.M



$$m_1\,\vec{r_1} + m_2\,\vec{r_2} = 0$$

$$\boxed{I_{AB} = I_{KL} + M.h^2}$$

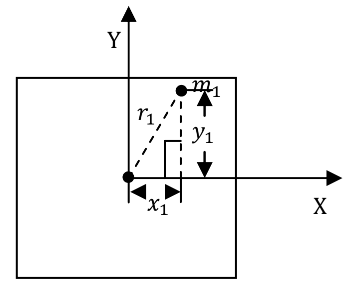
b) **Perpendicular Axis Theorem**:



$$\boxed{I_z = I_x + I_y}$$

Moment of Inertia about z-axis is sum of Moment of Inertia about x-axis and y-axis.

Proof:



Moment of Inertia about x-axis

$I_x = m_1\, y_1^2 + m_2\, y_2^2 + - - - - - - - + m_n\, y_n^2$

Moment of Inertia about y-axis

$I_y = m_1\, x_1^2 + m_2\, x_2^2 + - - - - - - - + m_n\, x_n^2$

Add both these equations

$I_x + I_y = m_1\,(x_1^2 + y_1^2) + m_2\,(x_2^2 + y_2^2) +$ ------------- $m_n\,(x_n^2 + y_n^2)$

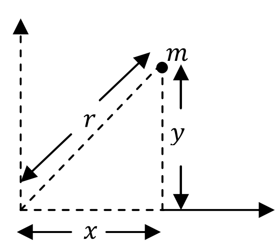
$= m_1\, r_1^2 + m_2\, r_2^2 +$ -------------- $m_n\, r_n^2$

$= \Sigma\, m\, r^2$

$= I_z$

$$\boxed{I_x + I_y = I_z}$$

Concept:



$r^2 = x^2 + y^2$

$I_x = m.\, y^2$

$I_y = m.\, x^2$

$I_x + I_y = m.\,(x^2 + y^2)$

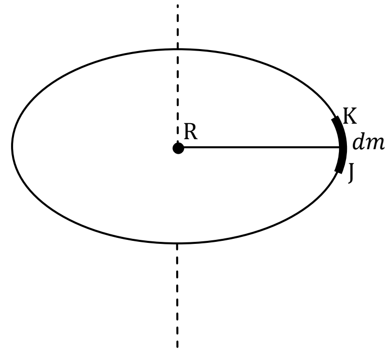
$= m.\, r^2$

$$\boxed{I_x + I_y = I_z}$$

Q4. Derive expression for moment of inertia of a circular ring and discuss all four possible cases.

Ans. **CASE I**:

- Moment of Inertia about an axis passing through centre and ⊥ to the plane.



Moment of Inertia of small part JK =

$dI = (dM)\, R^2$ ----------------- (1)

Moment of Inertia of ring, I =

Integrate both sides of eqt. (1)

$\int dI = \int (dM)\, R^2$

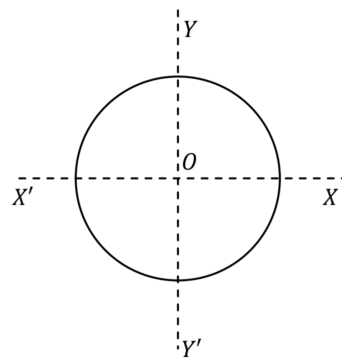
$I = R^2 \int (dM)$

$I = R^2\, M$

$\boxed{I = MR^2}$

CASE II:

- Moment of Inertia about diameter of ring



Moment of Inertia of ring about z-axis

$I_z = MR^2$ ----------------- (1)

Moment of Inertia of ring about x-axis

$I_x = I_{DIA}$ (?) ----------------- (2)

Moment of Inertia of ring about y-axis

$I_y = I_{DIA}$ (?) ----------------- (3)

As per ⊥ axis theorem

$I_z = I_x + I_y$

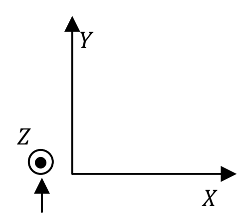
Put values from (1), (2) & (3)

$MR^2 = I_{DIA} + I_{DIA}$

$MR^2 = 2\, I_{DIA}$

$\frac{1}{2} MR^2 = I_{DIA}$

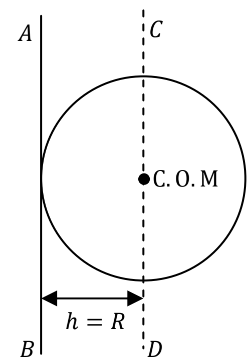
$\boxed{I_{DIA} = \frac{1}{2} MR^2}$



$z - axis\ is\ towards\ the\ reader$

CASE III:

- Moment of Inertia of uniform circular ring about a tangent in the plane of the ring



I_{AB} = ?

I_{CD} = I_{DIA} = $\frac{1}{2}MR^2$

As per || axis theorem

I_{AB} = $I_{CD} + M(h)^2$

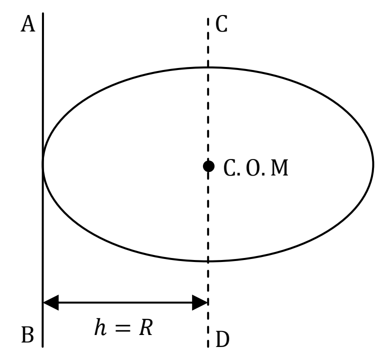
= $I_{CD} + M(R)^2$ [$h = R$]

= $\frac{1}{2}MR^2 + MR^2$

$\boxed{I_{AB} = \frac{3}{2}MR^2}$

CASE IV:

- Moment of Inertia of uniform circular ring about a tangent ⊥ to the plane of the ring



I_{AB} = ?

I_{CD} = MR^2

As per || axis theorem

I_{AB} = $I_{CD} + M(R)^2$

= $MR^2 + MR^2$

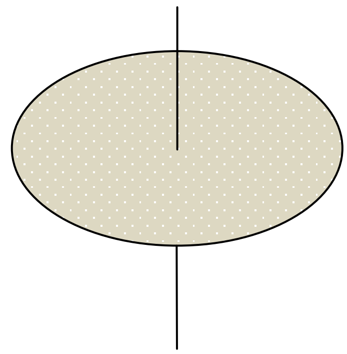
= $2\,MR^2$

$\boxed{I_{AB} = 2MR^2}$

Q5. Derive expressions for moment of inertia of uniform circular and discuss all four possible cases.

Ans. **CASE I**:

- Moment of Inertia about an axis passing through centre and ⊥ to the plane



Assume a strip of thickness dx at distance x (It is a ring), M.O.I of strip,

$dI = (dM) \cdot x^2$

Integrate both sides

$\int dI = \int (dM) \cdot x^2$

$I = \int (dM) \cdot x^2$

$I = \int \left(\frac{M}{\pi R^2} \cdot 2\pi x \cdot dx\right) x^2$

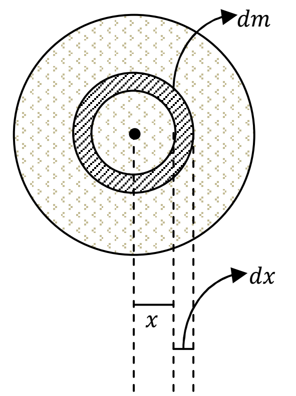
$I = \frac{2M}{R^2} \int x^3 \cdot dx$

$= \frac{2M}{R^2} \left|\frac{x^{3+1}}{3+1}\right|_{x=0}^{x=R}$

$= \frac{2M}{R^2} \cdot \frac{1}{4} (R^4 - 0^4)$

$= \frac{1}{2} \frac{M}{R^2} R^4$

$$\boxed{I = \frac{1}{2} MR^2}$$



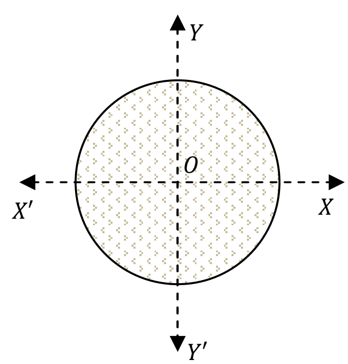
$dm = \left(\frac{M}{\pi R^2}\right)$ x (Area of strip)

$= \frac{M}{\pi R^2} (2\pi x \text{ x } dx)$

CASE II:

- Moment of Inertia about diameter

Moment of Inertia of disc about z-axis



$I_Z = \frac{1}{2} MR^2$ ----------------- (1)

Moment of Inertia of disc about x-axis

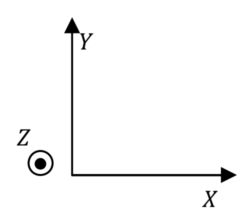
$I_X = I_{DIA}$ (?) ----------------- (2)

Moment of Inertia of disc about y-axis

$I_Y = I_{DIA}$ (?) ----------------- (3)

As per ⊥ axis theorem

$I_Z = I_X + I_Y$



Put value from (1), (2) & (3)

$\frac{1}{2} MR^2 = I_{DIA} + I_{DIA}$

$\frac{1}{2} MR^2 = 2I_{DIA}$ $\boxed{I_{DIA} = \frac{1}{4} MR^2}$

CASE III:

- Moment of Inertia of circular disc about a tangent in the plane of the disc.

I_{AB} = ?

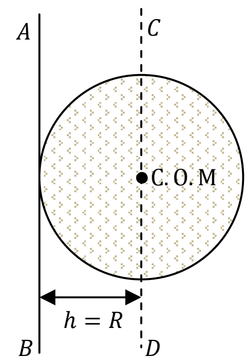
$I_{CD} = I_{DIA} = \frac{1}{4}MR^2$

As per || axis theorem

$I_{AB} = I_{CD} + M(R)^2$

$= \frac{1}{4}MR^2 + MR^2$

$$\boxed{I_{AB} = \frac{5}{4}MR^2}$$



CASE IV:

- Moment of Inertia of circular disc about a tangent ⊥ to the plane of the disc.

I_{AB} = ?

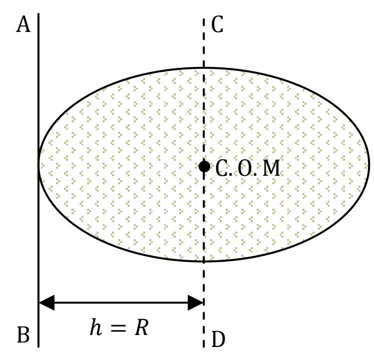
$I_{CD} = \frac{1}{2}MR^2$

As per || axis theorem

$I_{AB} = I_{CD} + M(R)^2$

$= \frac{1}{2}MR^2 + MR^2$

$$\boxed{I_{AB} = \frac{3}{4}MR^2}$$



Q6. An object rolls from height *h*, down an inclined plane. What fraction of K.E is rotational.

Ans. As object rolls down,

Decrease in P.E. = Increase in K.E.

$$mgh = K.E_{Trans} + K.E_{rotat}$$

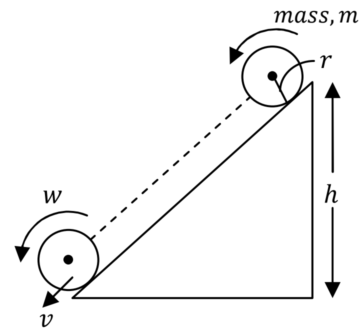
$$= \frac{1}{2} m v^2 + \frac{1}{2} . (\alpha m r^2) . w^2$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} \alpha m v^2$$

$$mgh = 1. \left(\frac{1}{2} m v^2\right) + \alpha \left(\frac{1}{2} m v^2\right)$$

Ratio $= 1 : \alpha$

$$\boxed{\frac{K.E_{rota}}{K.E_{Total}} = \frac{\alpha}{1+\alpha}}$$



$I = (\alpha m) R^2$

where $\alpha = 1$ for ring

$= \frac{1}{2}$ for disc

$= \frac{2}{5}$ for solid sphere

Q7. State and explain "Law of Conservation of Angular momentum"? give example.

Ans. Total angular momentum of the system remains constant if there is no external Torque acting on the system.

$$\tau_{ext} = 0$$

$$\frac{d(L)}{dt} = 0$$

$$\boxed{\begin{array}{l} L \rightarrow Constant \\ Iw \rightarrow Constant \end{array}}$$

Examples:

1. A *ballet dancer* rotating with her arms and legs stretched outwards. When she folds her arms and brings the stretched leg close to the other leg, moment of inertia decreases and hence her angular velocity increases.

2. The angular speed (*w*) of inner layers of the whirl wind (*tornado*) is alarmingly high. This is because moment of inertia (*I*) of inner layer is much smaller (*r is small*).

3. A diver performs somersaults by jumping from a high diving board keeping his legs and arms out stretched first and then curling his body. On doing so, the moment of inertia *I* of his body decreases, so angular velocity increases. Hence performs somersaults. As the diver is about to touch the surface of water, he stretches out his limbs. Moment of Inertia of his body increase and his angular velocity decrease.

4. All helicopters are provided with two propellers. If there were one single propeller, the helicopter would rotate itself in opposite direction in accordance with law of conservation of angular momentum.

5. Person carrying dumbles in his stretched hands and rotating. When he brings his arms holding dumbles closer to chest, Moment of Inertia decreases, and hence his angular velocity (*w*) increases.

6. While falling, a cat stretches its body along with its tail so that its Moment of Inertia increases. Hence, *w* angular velocity decreases and the cat lands gently on its feet.

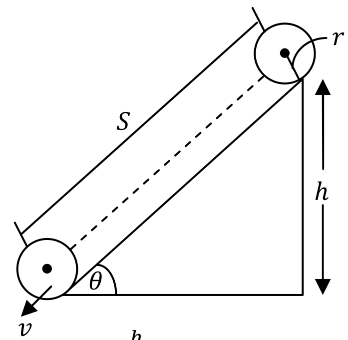
Q8. A solid cylinder rolls on an inclined plane (angle θ) from height *h*. Find

a) Final velocity

b) Acceleration

c) Time taken to reach bottom.

Ans.a) **<u>Final Velocity</u>**:



$\frac{h}{S}$ = sinθ

Decrease in P.E. $= \Delta K.E_{Trans} + \Delta K.E_{rotat}$

$= \frac{1}{2} m\, v^2 + \frac{1}{2} I\, w^2$

$= \frac{1}{2} m\, v^2 + \frac{1}{2} .(\alpha\, m\, r^2)\,.\, w^2$

$\cancel{m}gh = \left(\frac{1}{2}\cancel{m}\, v^2\right) + \alpha\left(\frac{1}{2}\cancel{m}\, v^2\right)$

$2gh = v^2 + \alpha\, v^2$

$2gh = (1 + \alpha)\; v^2$

$\frac{2gh}{1+\alpha} = v^2$

$\boxed{v = \sqrt{\frac{2gh}{1+\alpha}}}$ $\alpha = \frac{1}{2}$ for solid cylinder

- Velocity of object is more, if α is low.
- Some mass and size,
  velocity of sphere > velocity of disc > velocity of ring

b) **<u>Acceleration</u>**:

$v^2 - 0^2 = 2as$

$\frac{\cancel{2}g\cancel{h}}{1+\alpha} - 0^2 = \cancel{2}a\left(\frac{\cancel{h}}{sin\theta}\right)$

$\frac{g}{1+\alpha} = \frac{a}{sin\theta}$

$\boxed{\text{Acc.} = \frac{gsin\theta}{1+\alpha}}$ ($a = \frac{1}{2}$ solid cylinder)

c) **<u>Time</u>**:

$s = 0t + \frac{1}{2}a\, t^2$

$\frac{h}{sin\theta} = \frac{1}{2}\left(\frac{gsin\theta}{1+\alpha}\right) t^2$

$\frac{2h(1+\alpha)}{gsin\theta} = t^2$

$\boxed{t = \frac{1}{sin\theta}\sqrt{\frac{2h(1+\alpha)}{g}}}$ When θ & h same $\boxed{t \;\alpha\; \sqrt{(1 + \alpha)}}$

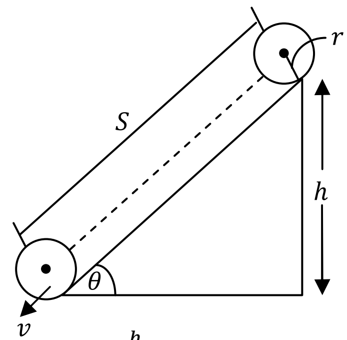
Q9. **A ring rolls on an inclined plane (angle θ) from height *h*. Find**

a) Final velocity

b) Acceleration

c) **Time taken to reach bottom.**

Ans. a) **<u>Final Velocity</u>**:



$\frac{h}{S} = \sin\theta$

Decrease in P.E. $= \Delta\, K.E_{Trans} + \Delta\, K.E_{rotat}$

$= \frac{1}{2} m\, v^2 + \frac{1}{2} I\, w^2$

$= \frac{1}{2} m\, v^2 + \frac{1}{2}.(\alpha\, m\, r^2)\,.\, w^2$

$\cancel{m}gh = \left(\frac{1}{2}\, \cancel{m}\, v^2\right) + \alpha \left(\frac{1}{2}\, \cancel{m}\, v^2\right)$

$2gh = v^2 + \alpha\, v^2$

$2gh = (1 + \alpha)\; v^2$

$\frac{2gh}{1 + \alpha} = v^2$

$v = \sqrt{\frac{2gh}{1 + \alpha}}$ (α = 1 for a ring)

$v = \sqrt{\frac{2gh}{2}}$ $v = \sqrt{gh}$

b) **<u>Acceleration</u>**:

$v^2 - 0^2 = 2as$

$g\cancel{h} - 0^2 = 2a\left(\frac{\cancel{h}}{\sin\theta}\right)$

$\frac{g\sin\theta}{2} = a$

c) **<u>Time</u>**:

$s = 0t + \frac{1}{2}\, a\, t^2$

$\frac{h}{\sin\theta} = \frac{1}{2}\left(\frac{g\sin\theta}{2}\right) t^2$

$\frac{4h}{g\sin\theta} = t^2$

$$\boxed{t = \frac{2}{\sin\theta}\sqrt{\frac{h}{g}}}$$

Q10. **A solid sphere rolls on an inclined plane (angle θ) from height *h*. Find**

a) Final velocity

b) Acceleration

c) Time taken to reach bottom.

Ans. a) **Final Velocity**:

Decrease in P.E. $= \Delta K.E_{Trans} + \Delta K.E_{rotat}$

$= \frac{1}{2} m\, v^2 + \frac{1}{2} I\, w^2$

$= \frac{1}{2} m\, v^2 + \frac{1}{2} .(\alpha\, m\, r^2)\, .\, w^2$

$\cancel{m}gh = \left(\frac{1}{2}\, \cancel{m}\, v^2\right) + \alpha \left(\frac{1}{2}\, \cancel{m}\, v^2\right)$

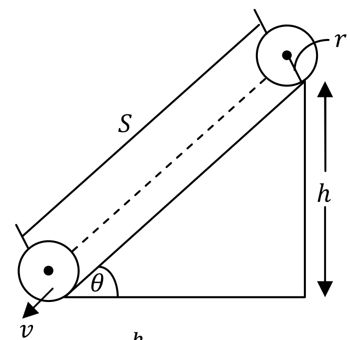
$2gh = (1 + \alpha)\; v^2$

$v = \sqrt{\frac{2gh}{1+\alpha}}$

$v = \sqrt{\frac{2gh}{1+\frac{2}{5}}}$

$v = \sqrt{\frac{2gh}{\frac{7}{5}}}$

$$\boxed{v = \sqrt{\frac{10gh}{7}}}$$



$\frac{h}{S} = \sin\theta$

$\alpha = \frac{2}{5}$ for solid sphere

b) **Acceleration**:

$v^2 - 0^2 = 2as$

$\frac{\cancel{2}g\cancel{h}}{1+\alpha} - 0^2 = \cancel{2}a \left(\frac{\cancel{h}}{sin\theta}\right)$

$\frac{g}{7/5} = \frac{a}{sin\theta}$

Acc $= \frac{gsin\theta}{7/5}$

$$\boxed{\text{Acc, } a = \frac{5gsin\theta}{7}}$$

c) **Time**:

$s = 0t + \frac{1}{2}\, a\, t^2$

$\frac{h}{sin\theta} = \frac{1}{2}\left(\frac{5gsin\theta}{7}\right) t^2$

$\frac{14h}{5gsin\theta} = t^2$

$$\boxed{t = \frac{1}{sin\theta}\sqrt{\frac{14h}{5g}}}$$